	z	x_1	x_2	x_3	x_4	RHS
z	1	−2	0	−3	0	−12
x_4	0	−3	0	−2	1	0
x_2	0	1	1	1	0	4

To find the range $[2, \lambda_2]$ over which this tableau is optimal, we first find $\bar{\mathbf{b}}$ and $\bar{\mathbf{b}}'$:

$$\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \end{bmatrix}$$

$$\bar{\mathbf{b}}' = \mathbf{B}^{-1}\mathbf{b}' = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Therefore, $S = \{2\}$ and λ_2 is given by

$$\lambda_2 = \frac{\bar{b}_2}{-\bar{b}_2'} = \frac{6}{-(-1)} = 6.$$

For λ in the interval [2, 6] the optimal objective value and the right–hand–side are given by

$$\begin{aligned} z(\lambda) &= \mathbf{c}_B\bar{\mathbf{b}} + \lambda\mathbf{c}_B\bar{\mathbf{b}}' \\ &= (0,-3)\begin{pmatrix} -6 \\ 6 \end{pmatrix} + \lambda(0,-3)\begin{pmatrix} 3 \\ -1 \end{pmatrix} = -18 + 3\lambda \\ \bar{\mathbf{b}} + \lambda\bar{\mathbf{b}}' &= \begin{bmatrix} -6 \\ 6 \end{bmatrix} + \lambda\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -6+3\lambda \\ 6-\lambda \end{bmatrix}. \end{aligned}$$

The optimal tableau over the interval [2, 6] is depicted below:

	z	x_1	x_2	x_3	x_4	RHS
z	1	−2	0	−3	0	$-18 + 3\lambda$
x_4	0	−3	0	−2	1	$-6 + 3\lambda$
x_2	0	1	1	1	0	$6 - \lambda$

At $\lambda = 6$, x_2 drops to zero. Since all entries in the x_2 row are nonnegative, we stop with the conclusion that no feasible solutions exist for all $\lambda > 6$. Figure 6.8 summarizes the optimal bases and the corresponding objective values for $\lambda \geq 0$. Note that the optimal objective value as a function of λ is piecewise linear and convex. In Exercise 6.66 we ask the reader to show that this is always true. The breakpoints correspond to the values of λ for which alternative optimal dual solutions exist.

Comment on Deriving Shadow Prices via a Parametric Analysis

Observe that parametric analysis can be used to ascertain the structure of the optimal value function $z^*(b_i)$ (see Equation (6.4)) as a function of b_i, in the neighborhood of the current value of b_i, for any $i \in \{1,\ldots,m\}$. Accordingly, we can then determine the right–hand and left–hand shadow prices with respect to

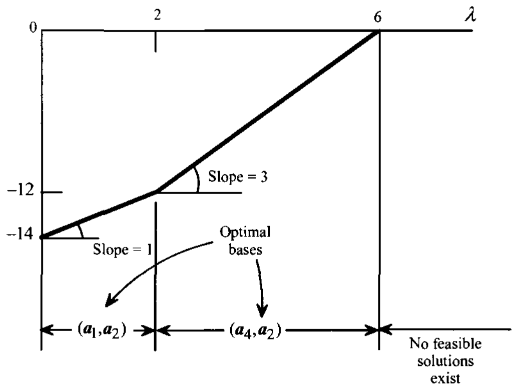

Figure 6.8. Optimal objectives and bases as a function of λ.

b_i as the respective right–hand and left–hand derivatives of $z^*(b_i)$ at the current value of b_i, where the former value is taken as infinity in case an increase in b_i renders the primal problem in Equation (6.1) infeasible.

More specifically, consider determining the right–hand shadow price with respect to b_i, for some $i \in \{1,\ldots,m\}$. In this case, the right–hand–side **b** is replaced by $\mathbf{b} + \lambda\mathbf{b}'$, where $\mathbf{b}' = \mathbf{e}_i$, the ith unit vector. Accordingly, we can now perform the foregoing parametric analysis until we arrive at a tableau that remains optimal as λ increases from zero (up to some positive level) or else, we detect unboundedness of the dual (infeasibility of the primal) as λ increases from zero. In the former case, the right–hand shadow price is given by $w_i = (\mathbf{c}_B\mathbf{B}^{-1})_i$ for the corresponding current tableau, and in the latter case, it is infinite in value. In a similar manner, we can compute the left–hand shadow price as the w_i value corresponding to the tableau that remains optimal as λ increases from the value of zero, where the right–hand–side is now perturbed according to $\mathbf{b} - \lambda\mathbf{e}_i$. Exercise 6.70 asks the reader to illustrate this approach.

EXERCISES

[6.1] Use the standard form of duality to obtain the dual of the following problem. Also verify the relationships in Table 6.1.

$$
\begin{array}{llllllll}
\text{Minimize} & \mathbf{c}_1\mathbf{x}_1 & + & \mathbf{c}_2\mathbf{x}_2 & + & \mathbf{c}_3\mathbf{x}_3 & & \\
\text{subject to} & \mathbf{A}_{11}\mathbf{x}_1 & + & \mathbf{A}_{12}\mathbf{x}_2 & + & \mathbf{A}_{13}\mathbf{x}_3 & \geq & \mathbf{b}_1 \\
 & \mathbf{A}_{21}\mathbf{x}_1 & + & \mathbf{A}_{22}\mathbf{x}_2 & + & \mathbf{A}_{23}\mathbf{x}_3 & \leq & \mathbf{b}_2 \\
 & \mathbf{A}_{31}\mathbf{x}_1 & + & \mathbf{A}_{32}\mathbf{x}_2 & + & \mathbf{A}_{33}\mathbf{x}_3 & = & \mathbf{b}_3 \\
 & & & & & \mathbf{x}_1 & \geq & \mathbf{0} \\
 & & & & & \mathbf{x}_2 & \leq & \mathbf{0} \\
 & & & & & \mathbf{x}_3 & & \text{unrestricted.}
\end{array}
$$

[6.2] Give the dual of the following problem:

$$\begin{array}{lrrrrrl} \text{Maximize} & -2x_1 & + & 3x_2 & + & 5x_3 & \\ \text{subject to} & -2x_1 & + & x_2 & + & 3x_3 & + x_4 \geq 5 \\ & 2x_1 & & & + & x_3 & = 4 \\ & & & -2x_2 & + & x_3 & + x_4 \leq 6 \\ & & & & & & x_1 \leq 0 \\ & & & & & x_2, & x_3 \geq 0 \\ & & & & & & x_4 \quad \text{unrestricted.} \end{array}$$

[6.3] Consider the following problem:

$$\begin{array}{lrrrl} \text{Maximize} & -x_1 & + & 3x_2 & \\ \text{subject to} & 2x_1 & + & 3x_2 & \leq 6 \\ & x_1 & - & 3x_2 & \geq -3 \\ & x_1, & & x_2 & \geq 0. \end{array}$$

a. Solve the problem graphically.
b. State the dual and solve it graphically. Utilize the theorems of duality to obtain the values of all the primal variables from the optimal dual solution.

[6.4] Solve the following linear program by a graphical method:

$$\begin{array}{lrrrrrl} \text{Maximize} & 3x_1 & + & 3x_2 & + & 21x_3 & \\ \text{subject to} & 6x_1 & + & 9x_2 & + & 25x_3 & \leq 15 \\ & 3x_1 & + & 2x_2 & + & 25x_3 & \leq 20 \\ & x_1, & & x_2, & & x_3 & \geq 0. \end{array}$$

(*Hint*: Utilize the dual problem.)

[6.5] Consider the following problem:

$$\begin{array}{lrrrrrrrrrl} \text{Maximize} & 10x_1 & + & 24x_2 & + & 20x_3 & + & 20x_4 & + & 25x_5 & \\ \text{subject to} & x_1 & + & x_2 & + & 2x_3 & + & 3x_4 & + & 5x_5 & \leq 19 \\ & 2x_1 & + & 4x_2 & + & 3x_3 & + & 2x_4 & + & x_5 & \leq 57 \\ & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq 0. \end{array}$$

a. Write the dual problem and verify that $(w_1, w_2) = (4, 5)$ is a feasible solution.
b. Use the information in Part (a) to derive an optimal solution to both the primal and the dual problems.

[6.6] Consider the following problem:

$$\begin{array}{lrrrrrrrl} \text{Minimize} & 2x_1 & + & 15x_2 & + & 5x_3 & + & 6x_4 & \\ \text{subject to} & x_1 & + & 6x_2 & + & 3x_3 & + & x_4 & \geq 2 \\ & -2x_1 & + & 5x_2 & - & 4x_3 & + & 3x_4 & \leq -3 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq 0. \end{array}$$

a. Give the dual linear problem.
b. Solve the dual geometrically.
c. Utilize information about the dual linear program and the theorems of duality to solve the primal problem.

[6.7] Consider the following linear programming problem:

$$\begin{array}{lrrrrrl} \text{Maximize} & 2x_1 & + & 3x_2 & + & 5x_3 & \\ \text{subject to} & x_1 & + & 2x_2 & + & 3x_3 & \leq 8 \\ & x_1 & - & 2x_2 & + & 2x_3 & \leq 6 \\ & x_1, & & x_2, & & x_3 & \geq 0. \end{array}$$